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n-Semimetrics

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We introduce n-semimetrics as a common extension of n-metrics and certain recent applied notions like the 3-way distance. The n-semimetrics are totally symmetric maps from E^{n+1} into $\mathbb{R}_+$ satisfying the simplex inequality, a direct extension of the common triangle inequality. Among the examples, we study in detail certain n-semimetrics on $\{0, 1\}^m$. We give a few constructions and extend the 2-way distances.

1. Introduction

In this paper we study n-semimetrics and their discrete aspects. They generalize n-metrics, which directly extend the usual metrics, having been around for 70 years and whose 1990 bibliography lists over 260 items [8]. The motivation for n-metrics came mostly from geometry and so the main interest has been in their topological aspects. Independently of n-metrics, papers have recently appeared on weaker versions of 2-metrics motivated by various discrete applications, for example, in statistics see [4, 9, 10]. Their common trait seems to be the total symmetry and the simplex inequality. The strong n-metrics axioms $d(x_1, x_1, x_2, \dots, x_n) = 0$ and the existence of x_{n+1} such that $d(x_1, \dots, x_{n+1}) > 0$ whenever $x_1, \dots, x_n$ are distinct, which make perfect sense in geometry, are too restrictive for applications and hence are substantially weakened in various ways. For this reason we decided to not postulate them at all and to only assume the total symmetry and simplex inequality.

In Section 2 we define n-semimetrics. To illustrate them, in Section 3 we study in detail some examples of n-semimetrics on $\{0, 1\}^m$. They can be naturally expressed in terms of the subsets of $M = \{1, \dots, m\}$. In particular, we completely characterize the n-semimetrics ∇^K_{mn} on $\mathcal{P}(M)$ which for any $a_1, \dots, a_{n+1} \in \mathcal{P}(M)$ count the number of elements of M that belong to exactly i sets $a_1, \dots, a_{n+1}$ for some $i \in K$ where K is a given family of pairwise disjoint intervals in $[1, n+1]$. We give a few other examples and constructions of n-semimetrics.

In Section 4 we extend the two-way distances from [10] to n-way distances. Here, $d(x_1, \dots, x_n)$ depends only on the pairwise distinct elements amongst $x_1, \dots, x_n$ (and not on their frequencies), satisfies $d(x, x, \dots, x) = 0$ and a stronger version of the simplex inequality. We show an up-construction of an $(n+1)$-way distance from an n-way distance and a down-construction of a weaker version of an $(n-1)$-way distance from an n-way distance. Finally, we give such a down-construction from an n-way distance to an $(n-1)$-semimetric provided $n > 1$ and X is finite.

The purpose of this paper is to draw attention to n-semimetrics. Many questions arise. One question is how to classify them, in particular for X finite (e.g., in L_1-n-semimetrics or n-hypersemimetrics). Obviously the n-semimetrics on X finite form a cone (Fact 1). What are its facets and rays? The calculations will be rather complex due to the high dimension of the cone (even for small $|X|$); moreover, matrices can no longer be used. A version of covariant maps for n-semimetrics will be the topic of another paper.

2. Definitions

Recall that a *metric* or a *metric space* is a pair (E, d) where E is a nonvoid set and $d : E^2 \longrightarrow \mathbb{R}_+$(the set of nonnegative reals) satisfies for all $x, y, z \in E$:

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(d1) $d(x, y) = 0 \iff x = y$,
(d2) $d(x, y) = d(y, x)$ (symmetry),
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality).

A basic example is $(\mathbb{R}^2, d)$ where d is the Euclidean distance of x and y; i.e., the length of the segment joining x and y. An immediate extension is $(\mathbb{R}^3, d)$ where $d(x, y, z)$ is the area of the triangle with vertices x, y and z. This leads to the following definition [2, 6, 7, 11]: a 2-*metric* is a pair (E, d) where E is a nonempty set and $d : E^3 \longrightarrow \mathbb{R}_+$ satisfies for all $x, y, z, t \in E$:

(d1′) $d(x, x, y) = 0$,
(d1″) $x \neq y \Longrightarrow d(x, y, u) > 0$ for some $u \in E$,
(d2) $d(x, y, z)$ is totally symmetric,
(d3) $d(x, y, z) \leq d(t, y, z) + d(x, t, z) + d(x, y, t)$ (the tetrahedron inequality).

The axiom (d2) means that the value of $d(x, y, z)$ is independent of the order of x, y and z. The axiom (d3) captures the fact that in $\mathbb{R}^3$ the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining three faces. Next, (d1′) states that certain degenerate triangles have area 0 while (d1″) stipulates that each pair of distinct points is on at least one nondegenerate triangle.

A 2-metric allows the introduction of several geometrical and topological concepts—e.g., the betweenness, convexity, line and neighborhood—which lead to interesting results.

For finite 2-metrics, for their polyhedral aspects and for applications, the axioms (d1′) and (d1″) seem to be too restrictive and so we drop them. The definition given below is formulated for an arbitrary positive integer n. A map $d : E^{n+1} \longrightarrow \mathbb{R}_+$ is *totally symmetric* if for all $x_1, \ldots, x_{n+1} \in E$ and every permutation π of $\{1, \ldots, n+1\}$

$$d(x_{\pi(1)}, \ldots, x_{\pi(n+1)}) = d(x_1, \ldots, x_{n+1}).$$

DEFINITION. Let $n > 0$. An *n-semimetric* is a pair (E, d) where $d : E^{n+1} \longrightarrow \mathbb{R}_+$ is totally symmetric and satisfies the *simplex inequality*: For all $x_1, \ldots, x_{n+2} \in E$

$$d(x_1, \ldots, x_{n+1}) \leq \sum_{i=1}^{n+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}). \tag{1}$$

Note the following immediate fact.

FACT 1. If (E, d) and (E, d') are n-semimetrics and $a, b \in \mathbb{R}_+$, then $(E, ad + bd')$ is an n-semimetric.

Here, as usual, for all $x_1, \ldots, x_{n+1} \in E$

$$(ad + bd')(x_1, \ldots, x_{n+1}) := ad(x_1, \ldots, x_{n+1}) + bd'(x_1, \ldots, x_{n+1}).$$

We give a few examples of n-semimetrics in the next section.

3. EXAMPLES

Given the importance of metric spaces on $\{0, 1\}^m$ in the theory of finite metric spaces, we consider n-semimetrics on $\{0, 1\}^m$. It will be more convenient to present them as n-semimetrics on the set $\mathcal{P}(M)$ of all subsets of $M := \{1, \ldots, m\}$.

For sets a and b the *symmetric difference* is $a \triangle b := (a \setminus b) \cup (b \setminus a)$. The cardinality of a set a is denoted by $|a|$.

EXAMPLE 1. For $a, b, c \subseteq M$ set

$$\delta(a, b, c) := \frac{1}{2}(|a \triangle b| + |a \triangle c| + |b \triangle c|). \tag{2}$$

We check that $(\mathcal{P}(M), \delta)$ is a 2-semimetric. Clearly δ is totally symmetric. To check (1), let $a, b, c, d \subseteq M$. From (2)

$$\delta(b, c, d) + \delta(a, c, d) + \delta(a, b, d) = |a \triangle d| + |b \triangle d| + |c \triangle d| + \delta(a, b, c) \geq \delta(a, b, c). \tag{3}$$

Clearly (3) holds with equality if and only if $a = b = c = d$. Moreover, $\delta(a, a, b) = 0 \iff a = b$ and therefore δ does not satisfies (d1$'$). Clearly if $m > 2$, the axiom (d1$''$) holds for δ.

If we represent each $a \subseteq M$ by its characteristic zero-one m-vector $\chi_a = (a_1, \dots, a_m)$ (with $a_i = 1$ if $i \in a$ and $a_i = 0$ if $i \notin a$), then

$$|a \triangle b| = |a_1 - b_1| + \cdots + |a_m - b_m|$$

is the Hamming distance of χ_a and χ_b. For this reason we could refer to δ as the *half-perimeter* 2-*semimetric*. This construction has the following immediate extension.

FACT 2. Let $n > 1$ and let $d : E^n \longrightarrow \mathbb{R}_+$ be totally symmetric. Define the perimeter map d^* by setting for all $e_1, \dots, e_{n+1} \in E$

$$d^*(e_1, \dots, e_{n+1}) := \frac{1}{n} \sum_{i=1}^{n+1} d(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1}). \tag{4}$$

Then d^* is an n-semimetric.

PROOF. Clearly d^* is totally symmetric. To prove the simplicial inequality let $e_1, \dots, e_{n+2} \in E$. For all $1 \leq j \leq n+1$ from (4)

$$\frac{1}{n} d(e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n+2}) \leq d^*(e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n+2})$$

and so by (4)

$$d^*(e_1, \dots, e_{n+1}) \leq \sum_{j=1}^{n+1} d^*(e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n+2}). \qquad \square$$

It can be verified that δ from (2) counts the number of elements that belong to exactly i sets amongst a, b, c where $1 \leq i \leq 2$. This is extended in the next lemma.

LEMMA 3. *Let* $m > 1$, $M := \{1, \dots, m\}$ *and*

$$1 \leq r \leq s \leq n+1, \qquad r \leq n, \qquad \tau := \min(n+1-r, s). \tag{5}$$

For all (not necessarily pairwise distinct) subsets $a_1, \dots, a_{n+1}$ *of* M *denote by* $\bigtriangledown_{mn}^{rs}(a_1, \dots, a_{n+1})$, *shortly by* $\bigtriangledown(a_1, \dots, a_{n+1})$, *the number of elements of* M *that belong to exactly* i *sets amongst* $a_1, \dots, a_{n+1}$ *for some* $r \leq i \leq s$. *Then:*

(i) for all $a_1, \dots, a_{n+1} \subseteq M$

$$\tau \bigtriangledown (a_1, \dots, a_{n+1}) \leq \sum_{i=1}^{n+1} \bigtriangledown(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+2}), \tag{6}$$

(ii) the coefficient τ in (6) *is the largest possible, and*

(iii) $\bigtriangledown$ is an n-semimetric on $\mathcal{P}(M)$.

PROOF. (i) Let $a_1, \ldots, a_{n-2} \subseteq M$. Let $x \in M$ belong to exactly t sets amongst $a_1, \ldots, a_{n+1}$. If $t < r$ or $t > s$, then x is not counted in $\bigtriangledown(a_1, \ldots, a_{n+1})$. Thus let $r \leq t \leq s$. We can arrange the notation so that

$$x \in (a_1 \cap \cdots \cap a_t) \setminus (a_{t+1} \cup \cdots \cup a_{n+1}). \tag{7}$$

For $i = 1, \ldots, n+1$ set

$$A_i := \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+2}\}, \qquad d_i := \bigtriangledown(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+2}). \tag{8}$$

We distinguish the following cases:
(1) Let $x \in a_{n+2}$. If $1 \leq i \leq t$, then in view of (7) and (8) the element x belongs to exactly t sets from A_i (namely $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_t, a_{n+2}$), while for $t < i \leq n+1$ the element x belongs to exactly $t+1$ sets from A_i (namely $a_1, \ldots, a_t, a_{n+2}$).

(a) Let $t < s$. Then by the definition of $\bigtriangledown$ the element x contributes 1 to d_i for all $i = 1, \ldots, n+2$ and hence it contributes $n+1$ to the right-hand side of (6). It contributes τ to its left-hand side whereby by hypothesis $\tau \leq s \leq n+1$.

(b) Thus let $t = s$. Then x contributes 1 to d_i exactly if $1 \leq i \leq t$; this is due to the fact that $t + 1 > t = s$ and therefore x contributes 0 to d_i for $i > t$. In this case the contribution of x to the left-hand side of (5) is τ and to its right-hand side is t whereby $\tau \leq s = t$.

(2) Thus let $x \notin a_{n+2}$. Note that for every $i = t+1, \ldots, n$, the element x belongs to exactly t sets from A_i (namely $a_1, \ldots, a_t$), while for $i = 1, \ldots, t$, it belongs to exactly $t-1$ sets from A_i (namely $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_t$).

(a) Let $r < t$. Again x contributes τ to the left-hand side of (6) and $n+1$ to the right-hand side whereby $\tau \leq s \leq n+1$.

(b) Let $t = r$. Then x contributes τ and $n+1-t$ to the left- and right-hand sides of (6) whereby $\tau \leq n+1-r \leq n+1-t$. This proves (i).

(ii) Choose

$$a_1 = \cdots = a_r := \{1\}, a_{r-1} = \cdots = a_{n+2} = \emptyset.$$

As $\bigtriangledown(a_1, \ldots, a_{n+1}) = \bigtriangledown(\{1\}, \ldots, \{1\}, \emptyset, \ldots, \emptyset) = 1$ while $d_i = 1$ for $r < i \leq n$ and $d_i = 0$ for $1 \leq i \leq r$, the value of τ in (6) satisfies $\tau \leq n+1-r$. Finally, choose $a_1 = \cdots = a_s = a_{n+2} := \{1\}$ and $a_{s+1} = \cdots = a_{n-1} := \emptyset$. Proceeding as above we obtain $\tau \leq s$. This shows that $\tau = \min(n+1-r, s)$ is the best possible and proves (ii).

(iii) From (5) clearly $s \geq 1$ and $n+1-r \geq 1$ whence $\tau \geq 1$ and the simplicial law follows from (6). $\square$

EXAMPLE 4. Consider $\bigtriangledown := \bigtriangledown_{mn}^{1n}$. Clearly for all subsets $a_1, \ldots, a_{n+1}$ of M

$$\bigtriangledown(a_1, \ldots, a_{n+1}) = |(a_1 \cup \cdots \cup a_{n+1}) \setminus (a_1 \cap \cdots \cap a_{n+1})| \tag{9}$$

counts the number of elements that belong to some a_i but not to all a_i. Note that $\tau = n$. From (9) it follows that

$$\bigtriangledown(a_1, \ldots, a_{n+1}) = 0 \iff a_1 = \cdots = a_{n+1};$$

whence the n-semimetric $\bigtriangledown$ satisfies only the weakest variant of (d1′). However, for all $a_1, \ldots, a_n \subseteq M$, there exists $u \subseteq M$ such that $\bigtriangledown(a_1, \ldots, a_n, u) > 0$ and so $\bigtriangledown$ satisfies the strongest variant of (d1″)

For a set A and a nonnegative integer k denote by $C(A,k)$ the family of all k-element subsets of A. Set $K := \{1,\ldots,n+1\}$. For $a_1,\ldots,a_{n+1} \subseteq M$ we have the following formula

$$\triangledown(a_1,\ldots,a_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i-1}\sum_{G\in C(K,i)}\left|\bigcap_{g\in G}a_g\right|$$

obtained by an inclusion–exclusion based on (9). For $n=1$ the metric $\triangledown$ is the Hamming distance $|a_1 \triangle a_2|$ mentioned in Example 2. The case $n=2$ was considered in [5, Example 3].

Let $\mathcal{C} \subseteq \mathcal{P}(M)$. In [1] Bassalygo introduced the following 'function of supports' of $\mathcal{C}$. For $0 < n \leq |\mathcal{C}|$ denote by s_n the least value of $\triangledown^{1,n-1}_{m,n-1}(a_1,\ldots,a_n)$ for pairwise distinct $a_1,\ldots,a_n \in \mathcal{C}$. We have $s_1 \leq s_2 \leq \cdots$. Let $d_1,\ldots,d_k$ denote the longest strictly increasing subsequence of $s_1, s_2,\ldots$. Bassalygo calls d_j the jth *generalized Hamming distance* of $\mathcal{C}$.

EXAMPLE 5. Consider $\triangledown := \triangledown^{11}_{mn}$ (i.e., $K = \{\{1\}\}$). Clearly for all subsets $a_1,\ldots,a_{n+1}$ of M the number $\triangledown(a_1,\ldots,a_{n+1})$ counts the number of elements of M that belong to exactly one of the sets $a_1,\ldots,a_{n+1}$. Note that $\tau=1$. For pairwise disjoint $a_1,\ldots,a_{n+1}$ and $a_{n+2} := a_1\cup\cdots\cup a_{n+1}$ the simplicial inequality is sharp. There is again an inclusion–exclusion formula

$$\triangledown(a_1,\ldots,a_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i-1}\sum_{G\in C(K,i)}\left|\bigcap_{g\in G}a_g\right|$$

(see e.g., [13, Section 1.3]). Again $\triangledown$ was considered in [5, Example 2].

The next example settles the case $r=s=n+1$ not covered by Lemma 3.

EXAMPLE 6. $\triangledown := \triangledown^{n+1,n+1}_{mn}$ is not an n-semimetric.

Indeed, $\triangledown(a_1,\ldots,a_{n+1}) = |a_1\cap\cdots\cap a_{n+1}|$. For $a_1=\cdots=a_{n+1}=\{1\}$ and $a_{n+2}=\emptyset$, the simplicial inequality becomes $1\leq 0$.

We extend Lemma 3 to a family of disjoint intervals in $\{1,\ldots,n+1\}$.

PROPOSITION 7. *Let $m>1$, $M := \{1,\ldots,m\}$ and let*

$$K := \bigcup_{i=1}^{k}\{r_i, r_i+1,\ldots,s_i\},$$

where

$$1 \leq r_1 \leq s_1 < \cdots < r_k \leq s_k \leq n+1, \qquad r_k \leq n.$$

For all (not necessarily distinct) subsets $a_1,\ldots,a_{n+1}$ of M denote by $\triangledown^K_{mn}(a_1,\ \ldots,a_{n+1})$ (shortly by $\triangledown(a_1,\ldots,a_{n+1})$) the number of elements of M that belong to exactly i sets amongst $a_1,\ldots,a_{n+1}$ for some $i\in K$. Set $\tau := min(n+1-r_k, s_1)$. Then

(i) for all $a_1,\ldots,a_{n+2}\subseteq M$

$$\tau\ \triangledown(a_1,\ldots,a_{n+1}) \leq \sum_{i=1}^{n+1}\triangledown(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{n+2}), \tag{10}$$

(ii) the coefficient τ in (10) is the largest possible, and

(iii) $\triangledown$ is an n-semimetric on $\mathcal{P}(M)$.

PROOF. Clearly $\triangledown := \triangledown^{r_1s_1}_{mn} + \cdots + \triangledown^{r_ks_k}_{mn}$ and the statement follows from Lemma 3 and Fact 1. $\square$

REMARK. For $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+$ the same result holds for $\triangledown := \alpha_1 \triangledown_{mn}^{r_1 s_1} + \cdots + \alpha_k \triangledown_{mn}^{r_k s_k}$.

EXAMPLE 8. In Proposition 7 set $r_1 = s_1 = 1, r_2 = s_2 = 3, \ldots, r_k = s_k = 2k+1$ where $k := \lfloor \frac{1}{2} n \rfloor$. Then $\triangledown_{mn}^K(a_1, \ldots, a_{n+1})$ is the number of elements of M contained in precisely an odd number of sets amongst $a_1, \ldots, a_{n+1}$. The n-semimetric $\triangledown_{mn}^K$ has been introduced in [12]; the paper gives an inclusion–exclusion type formula for $\triangledown_{mn}^K$ and Bonferroni-type inequalities.

The following example extends an example from [3].

EXAMPLE 9. Let $X := \{1, \ldots, n+1\}$ and let $d : X^n \longrightarrow \{0, 1\}$ be defined by setting

$$d(x_1, \ldots, x_n) := \begin{cases} 0 & \text{if } x_i = x_j \text{ for some } 1 \le i < j \le n \\ & \text{or } \{x_1, \ldots, x_n\} = \{1, \ldots, n\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then d is an $(n-1)$-semimetric satisfying for all $x_1, \ldots, x_{n-1} \in X$ the axioms (d1$'$) $d(x_1, x_1, x_2, \ldots, x_{n-1}) = 0$ and (d1$''$) $d(x_1, \ldots, x_{n-1}, n+1) > 0$ whenever $x_1, \ldots, x_{n-1}$ are distinct.

Suppose to the contrary that the simplicial inequality does not hold. Then there exist $x_1, \ldots, x_{n+1} \in X$ such that $d(x_1, \ldots, x_n) = 1$, while for all $j = 1, \ldots, n$

$$d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) = 0. \tag{11}$$

These equalities imply that $Y := \{x_1, \ldots, x_n\}$ satisfies $|Y| = n$ and $n+1 \in Y$. Without loss of generality we can assume $Y = \{2, \ldots, n+1\}$ and $(x_1, \ldots, x_n) = (2, 3, \ldots, n+1)$. The definition and ($11_n$) show that $x_{n+1} = 1$. From (11_1) we obtain $d(1, 3, \ldots, n+1) = 0$ contrary to the definition of d. Thus d is an $(n-1)$-metric. The last statement follows from the definition.

EXAMPLE 10. Let d be an $(n-1)$-semimetric on X. Define $d' : X^{n+1} \longrightarrow \mathbb{R}_+$ by setting

$$d'(x_1, \ldots, x_{n+1}) := \begin{cases} \max\{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) : 1 \le i \le n+1\} \\ \qquad\qquad \text{if } x_1, \ldots, x_{n+1} \text{ are pairwise distinct,} \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Then d' is an n-semimetric on X.

To prove the simplicial inequality let $x_1, \ldots, x_{n+2} \in X$ be such that $\delta := d'(x_1, \ldots, x_{n+1}) > 0$. Then $\delta = d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$ for some $1 \le i \le n+1$ and so

$$\delta \le d'(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}) \le \sum_{j=1}^{n+2} d'(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+2}).$$

Let d be an n-semimetric on X. Following an idea from [7] for $r_1, \ldots, r_n \in X$ and $\varepsilon \in \mathbb{R}, \varepsilon > 0$ set

$$B_\varepsilon(r_1, \ldots, r_n) := \{x \in X : d(x, r_1, \ldots, r_n) < \varepsilon\}.$$

For $n = 1$ the set $B_\varepsilon(r)$ is the standard (open) ball with center r and radius ε. The sets $B_\varepsilon(r_1, \ldots, r_n)$ can be used as a subbase of a topology on X.

4. n-WAY METRICS

We extend the three-way distance from [10] to n-metrics. Let $n > 0$. A totally symmetric map $d : X^n \longrightarrow \mathbb{R}_+$ is a *weak n-way distance* if for all $x_1, \ldots, x_{n+1} \in X$

(a) $d(x_1, \ldots, x_1) = 0$ and

(b)

$$d(x_1, \ldots, x_n) \leq \sum_{i=2}^{n+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}). \tag{12}$$

Note that (b) is stronger than the simplicial inequality because the summation only starts at $i = 2$. Clearly every weak n-way distance determines an $(n-1)$-semimetric. A weak n-way distance d is an *n-way distance* if for all $x_1, \ldots, x_n \in X$

(c)

$$d(x_1, x_1, x_3, \ldots, x_n) = d(x_1, x_3, x_3, \ldots, x_n) \leq d(x_1, x_2, \ldots, x_n). \tag{13}$$

In view of the total symmetry, (13) implies that $d(x_1, \ldots, x_n)$ only depends on the k-element set $\{x_{i_1}, \ldots, x_{i_k}\}$ such that $\{x_1, \ldots, x_n\} = \{x_{i_1}, \ldots, x_{i_k}\}$ (where $1 \leq i_1 < \cdots < i_k \leq n$). The following extends a concept from [10].

EXAMPLE 11. Let $\alpha : X \longrightarrow \mathbb{R}_+$ and $n > 2$. The star n-distance $d_\alpha : X^n \longrightarrow \mathbb{R}_+$ is defined as follows. Let $x_1, \ldots, x_n \in X$ and let $0 \leq i_1 < \cdots < i_k \leq n$ be such that $|\{x_1, \ldots, x_n\}| = |\{x_{i_1}, \ldots, x_{i_k}\}| = k$. Set

$$d_\alpha(x_1, \ldots, x_n) := \begin{cases} \alpha(x_{i_1}) + \cdots + \alpha(x_{i_k}) & \text{if } k > 1, \\ 0 & \text{if } k = 1. \end{cases}$$

FACT 12. (1) The star n-distance d_α is an n-way distance.
(2) d_α satisfies for all $x_1, \ldots, x_{n+1} \in X$

$$(n-2) d_\alpha(x_1, \ldots, x_n) \leq \sum_{i=2}^{n} d_\alpha(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \tag{14}$$

with equality if and only if $\alpha(x_i) > 0$ implies both (a) $\alpha(x_{n+1}) = 0$ and (b) x_i appears only once amongst $x_1, \ldots, x_n$ $(i = 1, \ldots, n+1)$.

PROOF OF FACT (2). There is nothing to prove if $x_1 = \cdots = x_n$. Thus we may assume that the sequence $\langle x_1, \ldots, x_n \rangle$ is $\langle x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k \rangle$ where $x_1, \ldots, x_k$ are distinct and x_i appears with the frequency φ_i $(i = 1, \ldots, k)$. Suppose $\alpha(x_i) > 0$ for some $1 \leq i \leq k$. If $\varphi_i > 1$ or $\alpha(x_{n+1}) = \alpha(x_i)$, then $\alpha(x_i)$ appears in each $d_\alpha(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$ with $2 \leq j \leq n$ and hence $\alpha(x_i)$ appears $n-1$ times on the right-hand side of (14). If $\varphi_i = 1$ and $\alpha(x_{n+1}) \neq \alpha(x_i)$, then $\alpha(x_i)$ appears only $n-2$ times on the right-hand side of (14). For the equality in (14) we need the latter case and $\alpha(x_{n+1}) = 0$.
Both (a) and (c) follow from the definition of d_α and (b) from (14). □

Lemmas 13–15 extend results from [10].

LEMMA 13. *Let d be a weak n-way distance on X. Define $d' : X^{n+1} \longrightarrow \mathbb{R}_+$ by setting for all $x_1, \ldots, x_{n+1} \in X$*

$$d'(x_1, \ldots, x_{n+1}) := \sum_{i=1}^{n+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}). \tag{15}$$

Then d' is a weak $(n+1)$-way distance on X.

PROOF. Clearly d' is totally symmetric and satisfies (a). To prove (b) let $x_1, \ldots, x_{n+2} \in X$. Then from (15) and (12)

$$\begin{aligned} d'(x_1, \ldots, x_{n+1}) &= d(x_2, \ldots, x_{n+1}) + \sum_{i=2}^{n+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \\ &\leq \sum_{j=3}^{n+1} d(x_2, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{n+2}) \\ &\quad + \sum_{i=2}^{n+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \\ &\leq \sum_{j=3}^{n+1} d(x_2, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{n+2}) \\ &\quad +2 \sum_{2\leq k<l\leq n+1} d(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{n+2}) \\ &\leq \sum_{p=2}^{n+1} \left(\sum_{q=1}^{p-1} d(x_1, \ldots, x_{q-1}, x_{q+1}, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{n+2}) \right. \\ &\quad \left. + \sum_{q=p+1}^{n+2} d(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_{n+2}) \right) \\ &= \sum_{p=2}^{n+1} d'(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{n+2}). \qquad \square \end{aligned}$$

We can also construct an $(n-1)$-way distance from an n-way distance.

LEMMA 14. *Let $n > 2$ and let d be an n-way distance on X. For all $x_1, \ldots, x_{n-1} \in X$ set*

$$d'(x_1, \ldots, x_{n-1}) := d(x_1, x_1, x_2, \ldots, x_{n-1}).$$

Then d' is an $(n-1)$-way metric on X.

PROOF. (a) is obvious. To prove (b) let $x_1, \ldots, x_n \in X$. Then applying (c) and (b) (to $x_1, \ldots, x_{n+1}, x_{n+1}$) and (c)

$$\begin{aligned} d'(x_1, \ldots, x_n) = d(x_1, x_1, x_2, \ldots, x_n) &\leq d(x_1, \ldots, x_{n+1}) \\ &\leq \sum_{j=2}^{n+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}, x_{n+1}) \\ &= \sum_{j=2}^{n+1} d(x_1, x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) \\ &= \sum_{j=2}^{n+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}). \end{aligned}$$

To show (c) for d' let $x_1, \ldots, x_n \in X$. Then

$$\begin{aligned} d'(x_1, x_1, x_3, \ldots, x_n) = d(x_1, x_1, x_1, x_3, \ldots, x_n) &= d(x_1, x_1, x_3, x_3, \ldots, x_n) \\ &\leq d(x_1, x_1, x_2, \ldots, x_n) \end{aligned}$$

and so (c) holds for d'. $\square$

The next lemma holds only for X finite.

LEMMA 15. *Let $n > 1$ and let d be an n-way distance on a finite set X. For all $x_1, \dots, x_n \in X$ set*

$$d'(x_1, \dots, x_n) := \sum_{x \in X} d(x, x_1, \dots, x_n).$$

Then d' is an $(n-1)$-semimetric on X.

PROOF. To prove the simplicial inequality for d' let $x_1, \dots, x_{n+1} \in X$. Applying the definition and (b) we obtain

$$\begin{aligned} d'(x_1, \dots, x_n) &= \sum_{x \in X} d(x, x_1, \dots, x_n) \\ &\le \sum_{x \in X} \sum_{j=1}^{n} d(x, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) \\ &= \sum_{j=1}^{n} \sum_{x \in X} d(x, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) \\ &= \sum_{j=1}^{n} d'(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}). \end{aligned}$$

□

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